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Quantum Mechanics on a Noncommutative Brane in M(atrix) Theory

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Abstract

We consider the quantum mechanics of a particle on a noncommutative two-sphere with the coordinates obeying an $SU(2)$-algebra. The momentum operator can be constructed in terms of an $SU(2) \times SU(2)$-extension and the Heisenberg algebra recovered in the smooth manifold limit. Similar considerations apply to the more general $SU(n)$ case.

question of setting up the quantum mechanics of a particle on a brane configuration in the matrix model of M-theory [1, 2]. It is by now clear that the matrix model can successfully describe many of the expected features of M-theory. Smooth brane configurations and solutions of M-theory can be obtained in the large N-limit of appropriate $(N \times N)$-matrix configurations [2, 3]. Now, brane solutions in M(atrix) theory are examples of noncommutative manifolds, specifically those with an underlying Lie algebra structure. The relationship between the matrix description and M-theory and strings suggests that noncommutative manifolds with an underlying Lie algebra structure (or their specializations into cosets) would be the most interesting ones from a physical point of view. Therefore we shall focus on such manifolds, although one can, of course, consider the question of quantum mechanics on more general noncommutative manifolds as well.

There is, by now, an enormous number of papers dealing with noncommutative geometry. One line of development has to do with spectral actions and the use of the Dirac operator to characterize the manifold, motivated by quantum gravity [4]. Quantization of such actions has also been attempted [5]. The majority of recent papers deals with noncommutative manifolds with an underlying canonical structure and the construction of field theories on these spaces [6]. There has also been some recent work on manifolds with an underlying Lie algebra structure, including the definition of a star product and the construction of gauge fields which take values in the enveloping algebra [7]. The topic of the present paper fits within the general milieu of these ideas and investigations, but we also have a very specific theoretical context, namely, brane solutions in M(atrix) theory. If the world is a brane [8], and if it is realizable as a solution in M(atrix) theory, then the quantum mechanics of a particle on a brane is clearly of more than mathematical interest.

Consider a particular brane solution in M-theory, say, the noncommutative spherical membrane. In this case the brane has the topology of $S^2 \times S^1$, where the S^2 is the noncommutative part described by matrices and S^1 denotes the compactified 11-th dimension. The two-sphere is given in terms of three matrix coordinates which may be taken as

$$Q_a = \frac{r}{\sqrt{j(j+1)}}\; t_a \tag{1}$$

where r is a fixed number which is the radius of the sphere and $t_a,\ a = 1, 2, 3,$ are the generators of $SU(2)$ in the $(2j+1)$-dimensional matrix representation. As the dimension $(2j + 1) \to \infty$, we get a smooth manifold which is S^2. This limit can be very explicitly understood by the representation

$$\begin{aligned} T_+ &= T_1 + iT_2 = z^2\partial_z - Nz \\ T_- &= T_1 - iT_2 = -\partial_z \\ T_3 &= z\partial_z - \tfrac{1}{2}N \end{aligned} \tag{2}$$

A basis of states on which these act is given by $|\alpha\rangle,\ \alpha = 0, 1, ..., N$ with $\langle z|\alpha\rangle = 1, z, z^2, ..., z^N$. The inner product is given by

$$\langle g|f\rangle = (N+1)\int \frac{d^2z}{\pi}\frac{1}{(1+z\bar{z})^{N+2}}\;\bar{g}\; f \tag{3}$$

The matrix elements of T_a with the $(N+1)$ states $|\alpha\rangle$ give the standard matrix version of t_a, *viz.*, $(t_a)_{\alpha\beta} = \langle\alpha|T_a|\beta\rangle$. By partial integration, we can see that T_a can be replaced, in matrix elements, by

$$\begin{aligned} T_+ &= \lambda\phi + z^2\partial_z \\ T_- &= \lambda\bar{\phi} - \partial_z \\ T_3 &= \lambda\phi_3 + z\partial_z \end{aligned} \tag{4}$$

where $\lambda = \frac{1}{2}(N+2)$ and

$$\phi = \frac{2z}{1+z\bar{z}}, \qquad \bar{\phi} = \frac{2\bar{z}}{1+z\bar{z}}, \qquad \phi_3 = \frac{1-z\bar{z}}{1+z\bar{z}} \tag{5}$$

As $N \to \infty$, the $\lambda\phi$-terms in the above expressions for T_a dominate and we find $t_a \to \lambda\phi_a$. Thus $Q_a \to r\phi_a$, with $\phi_a\phi_a = 1$. The membrane is described by the continuous coordinates $z, \bar{z}$.

At finite N, the two-sphere is described by the $(N+1)$ states $|\alpha\rangle$ which may be thought of as approximating the sphere by $(N+1)$ points, none of which has sharply defined coordinates. Translations of $|\alpha\rangle$ can be achieved by the use of $T_\pm$. However, this is not what we want. As $N \to \infty$, $T_\pm$ go over to $\phi, \bar{\phi}$ and correspond to the mutually commuting coordinates $z, \bar{z}$. They do not play the role of momenta conjugate to those coordinates obeying the Heisenberg algebra. We need to identify the momenta which lead to the Heisenberg algebra as $N \to \infty$. Since the latter does not have finite dimensional matrix representations, it is also clear that we should expect a modified algebra at finite N. Ultimately, from the point of view of noncommutative spaces, one keeps N finite, the limit being taken only to show agreement with the smooth manifold limit.

The classical dynamics of a particle moving on a sphere gives a clue to the choice of a momentum operator or generator of translations. In the classical case, we may write the momentum as $P_a = (1/q^2)\epsilon_{abc}q_b J_c$, where q_a is the coordinate and J_c is the angular momentum operator, here taken as the fundamentally defined quantity. (If we reduce J_c in terms of q_b, and p_b conjugate to it, we find $P_a = (\delta_{ab} - q_a q_b/q^2)\, p_b$, which are the correct translation generators consistent with $q_a q_a = 1$.) Absorbing r into the definition of Q_a, a possible choice of P_a is then $-\frac{1}{2}(\lambda/Q^2)\epsilon_{abc}(Q_b J_c + J_c Q_b)$, where we have symmetrized Q_a, J_a to form a hermitian combination. The operators Q_a, J_a obey the algebra

$$[Q_a, Q_b] = \frac{i}{\lambda}\epsilon_{abc}Q_c \tag{6}$$

$$\begin{aligned} [J_a, Q_b] &= i\epsilon_{abc}Q_c \\ [J_a, J_b] &= i\epsilon_{abc}J_c \end{aligned} \tag{7}$$

Notice that

$$\begin{aligned} \tfrac{1}{2}\epsilon_{abc}(Q_b J_c + J_c Q_b) \qquad &= \epsilon_{abc}\left(Q_b J_c + \frac{i}{2}\epsilon_{cbk}Q_k\right) \\ &= \epsilon_{abc}Q_b J_c - iQ_a \\ &= \epsilon_{abc}Q_b(J_c - \lambda Q_c) \\ &= \lambda\epsilon_{abc}Q_b K_c \end{aligned} \tag{8}$$

where $\lambda K_a = J_a - \lambda Q_a$. Further

$$\begin{aligned}
[K_a, K_b] &= \frac{i}{\lambda}\epsilon_{abc}K_c \\
[K_a, Q_b] &= 0
\end{aligned} \tag{9}$$

Therefore, rather than starting with J_a, we might as well consider the mutually commuting $SU(2) \times SU(2)$-algebra of Q_a, K_a and define the momentum operator as

$$P_a = \lambda \frac{\epsilon_{abc}}{\sqrt{Q^2K^2}} K_b Q_c \tag{10}$$

where $Q^2 = Q_a Q_a,\ K^2 = K_a K_a$. Obviously, $[P_a, Q^2] = [P_a, K^2] = 0$ so that there is no ordering ambiguity in the definition of P_a. We have chosen to divide by the symmetric expression $\sqrt{Q^2K^2}$ eventhough the classical expression had q^2. As we shall see below, $Q^2 \approx K^2$ in the continuous manifold limit. Also the parameter λ will be related to Q^2, K^2 below. The commutation rules for P_a become

$$\begin{aligned}
[P_a, Q_b] \ &= \frac{i}{\sqrt{Q^2K^2}}\left[\delta_{ab} Q\cdot K - \tfrac{1}{2}(Q_a K_b + Q_b K_a) - \frac{\epsilon_{abc}P_c}{2\lambda}\right] \\
[P_a, P_b] \ &= i\epsilon_{abc}\frac{Q\cdot K}{Q^2K^2}\ J_c
\end{aligned} \tag{11}$$

$J_a = \lambda(Q_a + K_a)$ are the generators of the diagonal $SU(2)$ subgroup.

The smooth manifold limit can be understood by considering large representations for Q_a and K_a, and analyzing representations of the diagonal $SU(2)$ of J_a. Labelling the corresponding spins by lower case letters, we find $\lambda^2 Q^2 = q(q+1),\ \lambda^2 K^2 = k(k+1),\ J^2 = j(j+1)$ and $2\lambda^2 Q\cdot K = j(j+1) - q(q+1) - k(k+1)$. If we take q, k very large and the combined spin j to be small and fixed, and $\lambda^2 = \sqrt{Q^2K^2} \approx q(q+1) \approx k(k+1)$, we find that the algebra (6), (9), (11) reduces to

$$\begin{aligned}
[Q_a, Q_b] \ &\approx 0 \\
[P_a, Q_b] \ &\approx -i\left(\delta_{ab} - \frac{Q_a Q_b}{Q^2}\right) \\
[P_a, P_b] \ &\approx -i\epsilon_{abc}\frac{J_c}{Q^2}
\end{aligned} \tag{12}$$

We also have $Q_a Q_a \approx 1$. Equations (12) are the Heisenberg algebra restricted to a smooth two-sphere of unit radius. (Taking $\lambda^2 = \sqrt{Q^2K^2}/r^2$, we can get a radius equal to r.)

The emergence of the continuous coordinates and the large λ-expansion can be seen in more detail as follows. We write a general $SU(2)$-valued (2×2)-matrix as

$$g = \sqrt{1 - \frac{x^2}{r^2}}\ +\ i\vec{\sigma}\cdot\frac{\vec{x}}{r} \tag{13}$$

where σ_a are the Pauli matrices. We then find that

$$\begin{aligned}
g^{-1}\ dg &= i\ \frac{\sigma_a}{2}\ E_{ab}dx^b \\
dg\ g^{-1} &= i\ \frac{\sigma_a}{2}\ \tilde{E}_{ab}dx^b
\end{aligned} \tag{14}$$

where

$$\begin{aligned} E_{ab} &= \frac{2}{r^2}\left[-\epsilon_{abc}x_c + \frac{\delta_{ab}(r^2-x^2)+x_a x_b}{\sqrt{r^2-x^2}}\right] \\ \tilde{E}_{ab} &= E_{ba} \end{aligned} \tag{15}$$

The above equations define the frame fields on $SU(2)$. The inverses to E_{ab}, $\tilde{E}_{ab}$ are given by

$$\begin{aligned} E^{-1}_{ab} &= \tfrac{1}{2}\left(\epsilon_{abc}x_c + \delta_{ab}\sqrt{r^2-x^2}\right) \\ \tilde{E}^{-1}_{ab} &= E^{-1}_{ba} \end{aligned} \tag{16}$$

The quantities

$$\begin{aligned} Q'_a &= \; i\tilde{E}^{-1}_{ka}\frac{\partial}{\partial x_k} \\ K'_a &= -iE^{-1}_{ka}\frac{\partial}{\partial x_k} \end{aligned} \tag{17}$$

obey mutually commuting $SU(2)$ algebras. Further, since $[x_a, \sqrt{r^2-x^2}\partial_b]-[x_b, \sqrt{r^2-x^2}\partial_a] = 0$, we see that we can shift Q'_a by x_a and K'_a by $-x_a$ and still obtain the same algebra. In other words, we can define

$$\begin{aligned} Q_a &= \left[\;\; x_a + \frac{i}{2\lambda}\left(\epsilon_{abc}x_c + \delta_{ab}\sqrt{r^2-x^2}\right)\frac{\partial}{\partial x_b}\right] \\ K_a &= \left[-x_a + \frac{i}{2\lambda}\left(\epsilon_{abc}x_c - \delta_{ab}\sqrt{r^2-x^2}\right)\frac{\partial}{\partial x_b}\right] \\ J_a &= \lambda(Q_a+K_a) = -i\epsilon_{abc}x_b\frac{\partial}{\partial x_c} \end{aligned} \tag{18}$$

This is in a form suitable for the large λ-expansion for $SU(2)\times SU(2)$, with the combined total spin being small. As $\lambda\to\infty$, the x_a-terms are dominant in the expressions for $Q_a,\ K_a$ and we get $Q_a \to x_a,\ K_a\to -x_a$. The algebra (6), (9), (11) reduces to

$$\begin{aligned} [P_a, Q_b] &\approx -i\left(\delta_{ab} - \frac{x_a x_b}{x^2}\right) \\ [P_a, P_b] &\approx -i\epsilon_{abc}\frac{J_c}{x^2} \end{aligned} \tag{19}$$

$x^2 = x_a x_a = r^2$ is a constant in this limit. The ϕ's given in (5) are a particular parametrization of the x_a's subject to $x_a x_a$ being constant.

In taking the limit as above we have retained S^2-topology for the smooth manifold. It is important to realize that since we are dealing with Q's which obey a Lie algebra, Q^2 is fixed for any representation and hence we will not get a flat Heisenberg algebra. A way to obtain the flat space algebra would be to take the radius r to be very large and then restrict the operators to a small neighbourhood on the sphere. This will lead to a flat two-dimensional Heisenberg algebra as $r\to\infty$. For example, we can expand around $x_a = (0,0,r)$. It

is interesting to see how this works out directly in terms of the operators $Q_a,\ K_a$. The neighbourhood of $x_a = (0, 0, r)$ corresponds to Q_3 and $-K_3$ being large. Since $Q_3 \sim r$ and $\lambda \sim k/r$, we see that $[Q_1, Q_2] \sim ir^2/k$ and so, the commutativity of coordinates in the large k-limit requires that $r^2 \sim k^\delta$ with $\delta < 1$ as $k \to \infty$. On the other hand, we also have $[P_1, P_2] \sim 1/r^2$ and the vanishing of this requires $\delta > 0$. The simplest and symmetrical choice is to take $\delta = \frac{1}{2}$ or $r \sim k^{\frac{1}{4}}$. We define eigenstates of $Q_3,\ K_3$ by

$$
\begin{aligned}
K_3\ |m, n\rangle &= (-k + m)\ |m, n\rangle \\
Q_3\ |m, n\rangle &= (\ \ k - n)\ \ |m, n\rangle
\end{aligned}
\tag{20}
$$

Restricting to small neighbourhood of large Q_3, $-K_3$ means that the integers $m,\ n$ can be considered to be small compared to k. In this case, introducing raising and lowering operators $\alpha^\dagger,\ \alpha$ for n and $\beta^\dagger,\ \beta$ for m, we find

$$
\begin{aligned}
Q_1 &= \frac{r}{\sqrt{2k}}\ (\alpha + \alpha^\dagger) \\
Q_2 &= \frac{ir}{\sqrt{2k}}\ (\alpha^\dagger - \alpha) \\
P_1 &= -\frac{i}{r}\sqrt{\frac{k}{2}}\ (\alpha - \alpha^\dagger + \beta^\dagger - \beta) \\
P_2 &= -\frac{1}{r}\sqrt{\frac{k}{2}}\ (\alpha + \alpha^\dagger + \beta^\dagger + \beta)
\end{aligned}
\tag{21}
$$

where we can take $r = r_0 k^{\frac{1}{4}}$ and, as usual, $\alpha|m, n\rangle = \sqrt{n}\ |m, n{-}1\rangle$, $\beta|m, n\rangle = \sqrt{m}\ |m{-}1, n\rangle$, etc. The flat space Heisenberg algebra is now easily verified.

In the usual procedure of quantization, starting with a set of classical coordinates q_a, one introduces the momenta and the phase space, thereby doubling the number of variables. The quantum theory is then defined by one irreducible representation of the Heisenberg algebra. equivalent to the standard Schrödinger representation. The restriction to irreducibility is equivalent to the requirement that the wavefunctions depend only on half of the phase space variables, the coordinates q_a, for example. This is the so-called polarization condition. For a given wavefunction, this allows the determination of the momenta as $p_a\psi = -i(\partial\psi/\partial q_a)$.

In our case, starting with Q_a, obeying the $SU(2)$-algebra (6), we introduce the $SU(2) \times SU(2)$-algebra (6), (9) of $Q_a,\ K_a$. The set $Q_a,\ K_a$ can be considered as the analogue of the phase space. The analogue of the polarization condition implies that we must choose an irreducible representation of Q_a, K_a. As we have seen already, the smooth manifold limit corresponds to $q, k \to \infty$. For a given irreducible representation, labelled by the spin values (q, k), there are several representations possible for the angular momentum $J_a = \lambda(Q_a + K_a)$, the lowest possible j-value being $|q - k|$. The difference $|q - k|$ may be interpreted as the strength of a magnetic monopole at the center of the sphere, or a uniform magnetic field through the sphere. (I thank Polychronakos for discussions clarifying this point.) In the absence of any magnetic monopole field, we can take $q = k$.

Generalization to $SU(n)$

More general brane solutions require the consideration of N-dimensional representations of $SU(n)$, $n > 2$, with $N \to \infty$ eventually. The generalization of our considerations to $SU(n)$ is straightforward. Basically one has to consider an $SU(n) \times SU(n)$-algebra

$$\begin{aligned}
[Q_a, Q_b] &= \frac{i}{\lambda} f_{abc} Q_c \\
[K_a, K_b] &= \frac{i}{\lambda} f_{abc} K_c \\
[K_a, Q_b] &= 0
\end{aligned} \tag{22}$$

The momentum operator can then be defined by

$$P_a = \frac{n\lambda}{2}\ f_{abc} \frac{K_b Q_c}{\sqrt{Q^2 K^2}} \tag{23}$$

P_a is a derived quantity, with $Q_a,\ K_a$ defining the basic algebra, as in the case of $SU(2)$. The commutator of P_a with Q_b can be evaluated without too much trouble, eventhough it is more involved than in the case of $SU(2)$. The following identity for the the structure constants is useful for this calculation. Let t^a be hermitian $(n \times n)$-matrices which form a basis of the Lie algebra of $SU(n)$ with $[t^a, t^b] = i f^{abc} t^c$, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. We can then write

$$\begin{aligned}
f_{amc} f_{bkc} + f_{bmc} f_{akc} &= \frac{\partial^4}{\partial x^a \partial x^b \partial y^m \partial y^k}\ F \\
F &= -\mathrm{Tr}\left\{[t \cdot x, t \cdot y]\ [t \cdot x, t \cdot y]\right\}
\end{aligned} \tag{24}$$

The traces can be evaluated using the identity

$$t \cdot x\ t \cdot y + t \cdot y\ t \cdot x = \frac{x \cdot y}{n} + 2\ d_{abc} x_a y_b\ t_c \tag{25}$$

where $d_{abc} = \mathrm{Tr}\{(t_a t_b + t_b t_a) t_c\}$. Equation(24) then becomes the identity

$$\begin{aligned}
f_{amc} f_{bkc} + f_{bmc} f_{akc} &= \Big[\frac{4}{n} \delta_{ab} \delta_{mk} - \frac{2}{n} \left(\delta_{am} \delta_{bk} + \delta_{ak} \delta_{bm}\right) \\
&\qquad + \ 8\ d_{abc} d_{mkc}\ - 4\ d_{amc} d_{bkc}\ - 4\ d_{akc} d_{bmc}\Big]
\end{aligned} \tag{26}$$

With the help of this identity, the commutator of P_a with Q_b is now obtained as

$$\begin{aligned}
[P_a, Q_b] &= \frac{i}{\sqrt{K^2 Q^2}} \left[\delta_{ab} K \cdot Q - \tfrac{1}{2} \left(K_a Q_b + K_b Q_a\right)\right] \ + \ \frac{i}{2\lambda} f_{abc} P_c \\
&\qquad + \ \frac{in}{\sqrt{K^2 Q^2}}\ K_m Q_n\ \left(2\ d_{abc} d_{mnc}\ - d_{amc} d_{bnc} - d_{bmc} d_{anc}\right)
\end{aligned} \tag{27}$$

The calculation of $[P_a, P_b]$ is more involved. It does not seem to be very illuminating for our discussion.

It is also possible to develop expressions for $Q_a,\ K_a$, which are analogues of equations (13)-(18), in terms of an (n^2-1)-vector x_a which parametrizes $SU(n)$. We can write the

variation of a group element $g \in SU(n)$ as $g^{-1}\, dg = it_a E_{ab} dx^b$ and $dg\; g^{-1} = it_a \tilde{E}_{ab} dx^b$. The quantities E_{ab} and $\tilde{E}_{ab}$ are transposes of each other. For example, if we use an exponential parametrization $g = \exp(it\cdot x)$, we can write

$$\begin{aligned} E_{ab} &= \int_0^1 d\alpha\; 2\; \mathrm{Tr}\left(t_a e^{-i\alpha t\cdot x} t_b e^{i\alpha t\cdot x}\right) \\ \tilde{E}_{ab} &= \int_0^1 d\alpha\; 2\; \mathrm{Tr}\left(t_a e^{i\alpha t\cdot x} t_b e^{-i\alpha t\cdot x}\right) \end{aligned} \tag{28}$$

This equation shows explicitly that $E_{ab} = \tilde{E}_{ba}$. The left and right translation generators are then

$$\begin{aligned} L_a &= \; i\tilde{E}^{-1}_{ka} \frac{\partial}{\partial x^k} \\ R_a &= -iE^{-1}_{ka} \frac{\partial}{\partial x^k} \end{aligned} \tag{29}$$

with $L_a g = -t_a g$, $R_a g = g t_a$. These obey the Lie algebra relations $[\xi_a, \xi_b] = i f_{abc} \xi_c$, $\xi = L,\ R$. In terms of $V_a \equiv (L_a + R_a)$ and $A_a \equiv (L_a - R_a)$, this becomes

$$\begin{aligned} &[V_a, V_b] = i f_{abc} V_c \\ &[V_a, A_b] = i f_{abc} A_c \\ &[A_a, A_b] = i f_{abc} V_c \end{aligned} \tag{30}$$

Since A_a involves the symmetric combination $E^{-1}_{ka} + \tilde{E}^{-1}_{ka}$, the last of these relations is unaltered by shifting the A_a by x_a, i.e., $[A_a + x_a, A_b + x_b] = i f_{abc} V_c$. Further, $e^{i\theta\cdot V} g = e^{-it\cdot\theta} g e^{it\cdot\theta}$, showing that x_a transforms as a vector under the action of V_a. The operators Q_a and K_a can then be defined as

$$\begin{aligned} Q_a &= \;\; x_a \; + \; \frac{1}{2\lambda}(V_a + A_a) \\ K_a &= -x_a \; + \; \frac{1}{2\lambda}(V_a - A_a) \end{aligned} \tag{31}$$

These can be used as the starting pont for a large λ-expansion around some chosen value of x_a.

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